FINALS WEEK!
MATH 34A
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Drop-in Session: TBA
LAST UPDATED: 6:54PM, 12 December 2017

On this worksheet are selected problems from homeworks 9 and 10 which were less done. Many of these problems highlight a number of key concepts from class, and in my opinion, would make excellent exam questions. I recommend you guys to try doing these on your own before asking me for help or working with others.

As before, I do not have any part in writing the final exam, nor did the professor have any part in giving these problems. I would imagine, though, that if you knew how to do these questions, you would be in pretty good shape for the exam (that is, you did the questions yourself and understand how to do them).

Good luck on your finals!

6.31 A plane flies at 200 mph for the first and last half hour of a flight. It flies at 400 mph the rest of the time. The route is 1000 miles long. The town of Erehwon is on the route 300 miles before the end. What is the distance of the plane from Erehwon after t hours of flying?

(a) $t \leq \frac{1}{2}$

(b) $\frac{1}{2} \leq t \leq 2$

(c) $2 \leq t \leq 2\frac{1}{2}$

(d) $2\frac{1}{2} \leq t \leq 3$

7.52 A commuter railway has 800 passengers per day and charges each one two dollars per day. For each 1 cents that the fare is increased, 5 fewer people will go by train. What is the greatest profit that can be earned.

7.55 What point on the graph $y = \sqrt{x}$ is closest to $(10, 0)$. (Hint: work out the square of the distance of a point on the curve from $(10, 0)$ and minimize the distance squared, this makes the algebra easier).

First, we know that every point on the graph of $y = \sqrt{x}$ is of the form $(x, \sqrt{x})$. By the distance formula, we know that the distance of a point $(x, \sqrt{x})$ with $(10, 0)$ is $d = \sqrt{(10 - x)^2 + (0 - \sqrt{x})^2} = \sqrt{x^2 - 20x + 100 + x} = \sqrt{x^2 - 19x + 100}$.. Taking the hint, we know that the point $(x, \sqrt{x})$ that minimizes d is exactly the point that minimizes d^2 (which we refer to as D). So, let us minimize D.

We see that by definition, $D = x^2 - 19x + 100$. To minimize it, we take the derivative, set it equal to zero, and solve for x. We note that $D' = 2x - 19$, and by setting $2x - 19 = 0$, we see $x = \frac{19}{2}$.

So, we have found the x-coordinate of the point that minimizes D (which also minimizes d). We plug it into $y = \sqrt{x}$ to obtain the y-coordinate, which is $\sqrt{19/2}$. So, we found our point: $(\frac{19}{2}, \sqrt{\frac{19}{2}})$.

8.14 Beaker A contains 1 liter which is 20 percent oil and the rest is vinegar, thoroughly mixed up. Beaker B contains 2 liters which is 45 percent oil and the rest vinegar, completely mixed up. Half of the contents of B are poured into A, then completely mixed up. How much oil should now be added to A to produce a mixture which is 60 percent oil?

For problems like this, it is important to keep track of all your objects. So, let create a table for the amount of oil, vinegar, and total liquids.

Beaker	Oil	Vinegar	Total
A	?	?	1
B	?	?	2

Clearly, we need to fill in our table, so let's do that. We know that Beaker A has 20% oil, which means it's 80% vinegar. This means that there is $0.20 * 1$ liters of oil, and $0.80 * 1$ liters of vinegar. We can similarly calculate this for Beaker B, in which case, we get...

Beaker	Oil	Vinegar	Total
A	.2	.8	1
B	.9	1.1	2

Now, we are told that half the contents of B are poured into A. This tells us...

Beaker	Oil	Vinegar	Total
A	.65	1.35	2
B	.45	.55	1

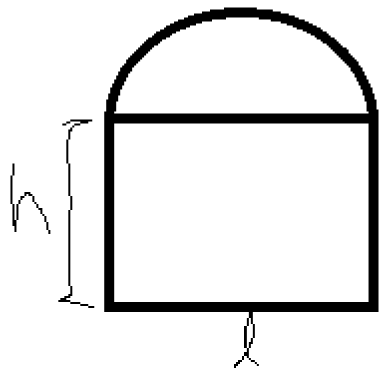
Now, we want to get 60% oil. We note that if we add x liters of oil, we get...

Beaker	Oil	Vinegar	Total
A	.65+x	1.35	2+x
B	.45	.55	1

Since we want the percent of oil in Beaker A to be 60%, we need $\frac{\text{amount of oil in Beaker A}}{\text{total volume in Beaker A}} = \frac{.65+x}{2+x} = 0.6$. Solving for x gets us $x = 1.375$, which is the amount of oil we need to add.

8.23 A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 50 ft, find the dimensions of the window so that the greatest possible amount of light is admitted. That is, find the base length and total height.

Our window looks something like this.



From above, we see that the area is the area of the square plus the area of the semicircle, which would be $A = \ell w + \frac{1}{2}\pi(\frac{\ell}{2})^2$ (the radius of the circle is $\ell/2$). We want to maximize this (as this would allow the most sunlight to come in!), but we have two variables, so let's get rid of one.

We know that the perimeter is 50 feet, which tells us that $50 = 2w + \ell + \pi(\frac{\ell}{2})$ (the perimeter counts the part of the rectangle not touching the semicircle, as well as the round part of the semicircle). Solving for w give us $w = \frac{50-\ell-\pi(\frac{\ell}{2})}{2}$.

We can now plug this into our area equation and obtain $A = \ell(\frac{50-\ell-\pi(\frac{\ell}{2})}{2}) + \frac{1}{2}\pi(\frac{\ell}{2})^2$, which can be simplified to $A = -\frac{\pi\ell^2}{8} - \frac{\ell^2}{2} + 25\ell$. We can now maximize this by taking the derivative, setting it equal to 0, and solving for ℓ. Doing so gives us that $\ell = \frac{200}{4+\pi}$, the base length that maximizes area. The total height is simply $h = w + \frac{\ell}{2}$, which we can be expressed as $\frac{50-\ell-\pi(\frac{\ell}{2})}{2} + \frac{\ell}{2}$. Plugging in what we found for ℓ gets us the total height when area is maximized.

8.24 A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 10 km/h and reaches the same dock at 3:00 P.M. How many minutes past 2:00 P.M. were the boats closest together?

We can envision all of this happening on the coordinate plane, with north being the positive x, south being the negative x, etc.

We can let the dock be the origin $(0,0)$, from which, we see that at time t (in hours!), the coordinates of the first boat is $(0,-20t)$. Since the second boat is traveling east, and reaches the dock at 3PM (an hour after 2PM), and is traveling 10 km/hr, we know that he starts off at $(-10,0)$ (ie. 10 km west of the dock). Since he is going $10km/hr$ east, his position at time t (hours!) is $(-10+10t,0)$.

By the distance formula, we know that at any time t, the distance between the two boats is $d=\sqrt{(-10+10t)^2+(-20t)^2}$. As we did so in 7.55, the point that minimizes d is also the point that minimizes $D=d^2=(-10+10t)^2+(-20t)^2=500t^2-200t+100$. To minimize D, we take the derivative, set it equal to 0, and solve for x (do this yourself!), and we get $t=\frac{1}{5}$. This means that the time in which the distance between the boats is minimized is $t/5$ hours after 2PM, or 12 minutes after 2PM.

9.27 A box has rectangular sides, top and bottom. The volume of the box is 3 cubic meters. The height of the box is half the width of the base. Express the total surface area of the box in terms of the height of the box.

If h is the height of the box, what is the surface area?

We know that given length l, width w, and height h, the volume of our box is given to be $V = lwh$. Right off the bat, we are told that $h = \frac{1}{2}w$ (so $w = 2h$) and $V = 3$. This tells us that $2lh^2 = 3$, so $l = \frac{3}{2h^2}$.

We know that the surface area is given to be $A = 2lw+2lh+2wh$. Plugging everything we got in from above, we have that $A = 2(\frac{3}{2h^2})(2h) + 2(\frac{3}{2h^2})h + 2(2h)h$, which is what we want.

9.28 A tank of water has a base a circle of radius 2 meters and vertical sides. If water leaves the tank at a rate of 6 liters per minute, how fast is the water level falling in centimeters per hour? [1 liter is 1000 cubic centimeters]

The difficulty in this problem is that everything is given in different units. So, let's convert everything into things with centimeters. 2 meters is 200 centimetrs, and 6 liters per minute becomes 6000 cubic centimeters per minute.

Now, let's set up our volume equation. From what it tells us, we see that our tank of water is a cylinder. Given a height of h and radius r, we know that the volume is $V = \pi r^2 h$. We note that r does not change, and is constantly 200 (centimeters, since we converted this from meters). So, we have that $V = 200^2 \pi h$. Now, applying $\frac{d}{dt}$ to both sides yields $\frac{dV}{dt} = 200^2 \pi \frac{dh}{dt}$. We want to know how fast the water is falling, given that water leaves 6 liters (ie. $6000 cm^3/min$). So we plug in 6000 into dV/dt, and obtain $6000 = 200^2 \pi \frac{dh}{dt}$. Solving for $\frac{dh}{dt}$ gets us that it is $\frac{6000}{200^2 \pi}$. However, this is in cm^3/min not hours! So, in order to convert to cm^3/hr, we multiply by 60, and get that the rate the height is changing is $60 \cdot \frac{6000}{200^2 \pi}$.

9.35 The manager of a large apartment complex knows from experience that 90 units will be occupied if the rent is 324 dollars per month. A market survey suggests that, on the average, one additional unit will remain vacant for each 1 dollar increase in rent. Similarly, one additional unit will be occupied for each 1 dollar decrease in rent. What rent should the manager charge to maximize revenue?

[This is the solution provided by WebWork]

As with most optimization problems, our goal is to write a formula for the value we are trying to optimize in terms of the value which can be changed. In this case, we need a formula for the revenue in terms of the amount of rent charged. First, note that the total revenue will be the number of rooms rented, which we will call R, times the amount of rent charged per room, which we will designate as x.

Note that we can write R in terms of x using the information from the problem. From the information in the problem, it is clear that the relationship between R and x is linear. One point on the line is given by $(90, 324)$ as we are told that 90 will be occupied if the rent is 324. Another point on the line is $(901, 324 + 1)$ since one additional unit will be vacant for each 1 dollar increase. Thus, using these two points, we can develope the equation for this line, $R = \frac{-1}{1}x + 414$

From this information, we now have a formula for revenue. Namely, $f(x) = R \times x = \frac{-1}{1}x^2 + 414x$. This is the function we wish to maximize, so we find its derivative and solve for zero. The derivative is given by $f'(x) = \frac{-2}{1}x + 414$. Solving for zero yields:

$$
\begin{aligned}
f'(x) &= 0 \\
\frac{-2}{1}x + 414 &= 0 \\
\frac{-2}{1}x &= -414 \\
x &= \frac{414(1)}{2} \\
x &= \frac{414}{2} \\
x &= 207
\end{aligned}
$$

Therefore, the maximum revenue occurs when we rent the units for 207 dollars each.

9.38 A manufacture has been selling 1750 television sets a week at \$510 each. A market survey indicates that for each \$13 rebate offered to a buyer, the number of sets sold will increase by 130 per week.

(a) Find the function representing the demand $p(x)$, where x is the number of the television sets sold per week and $p(x)$ is the corresponding price.

(b) How large rebate should the company offer to a buyer, in order to maximize its revenue?

(c) If the weekly cost function is $148750 + 170x$, how should it set the size of the rebate to maximize its profit?

10.3 A street light is at the top of a 19 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of 7 ft/sec along a straight path.

(a) How fast is the tip of her shadow moving along the ground when she is 45 ft from the base of the pole?

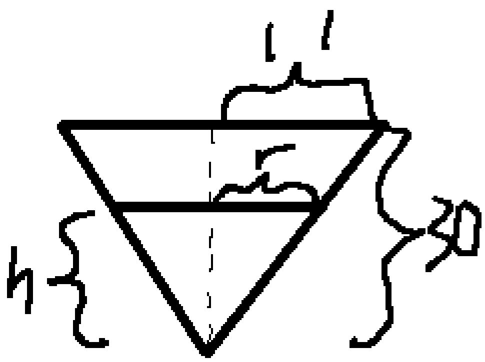
(b) How fast is the length of her shadow increasing?

10.5 A conical water tank with vertex down has a radius of 11 feet at the top and is 30 feet high. If water flows into the tank at a rate of 10ft^3/min, how fast is the depth of the water increasing when the water is 14 feet deep?

When the height of our water tank is h with a radius of r, we know that the volume of our water tank (which is a cone) is $V = \frac{1}{3}\pi r^2 h$. So can we just go ahead and differentiate? Not quite... we have two variables, both of which change as time goes on... So, let's get rid of one of them!

What the problem is asking us to do is to find the rate the height is changing. So, it makes sense to try expressing volume as a function of height. That is, we need to somehow replace the r in our volume equation with something that only has h.

Since all of this is taking place in a conical water tank with radius 11 and height 30, we have the following diagram as shown below:



We know that corresponding sides of similar triangles are proportional, which means we have that $\frac{30}{11} = \frac{h}{r}$, and thus, we get that $r = \frac{11}{30}h$. We can now plug that into our volume equation, thus getting $V = \pi \left(\frac{11}{30}h\right)^2 h = \frac{121\pi h^3}{2700}$.

Now, we can apply $\frac{d}{dt}$ to both sides, and we get that $\frac{dV}{dt} = \frac{121\pi h^2}{900}\frac{dh}{dt}$. We are asked to find the rate of change of height when $h = 14$ and $\frac{dV}{dt} = 10$, and so, we get $10 = \frac{121\pi(14)^2}{900}\frac{dh}{dt}$. Solving for $\frac{dh}{dt}$ gets us $\frac{dh}{dt} = \frac{900*10}{121*\pi*14^2} \approx 0.120796\cdots$

10.14 As a preparation for the long bright summer days, Dr. Acula plans to store gourmet plasma in closed tin cans that have the shape of a cylinder with volume V. As an environmentally conscious member of the community, he wants to use as little metal as possible.

(a) What is the height $h(r)$ for the can with minimum surface area in terms of the radius r of the bottom?

(b) What is the minimum surface area for the can in terms of the portion size V?

(a) To be honest, this is a pretty strange and confusing problem, in that it's not obvious what the question is asking. Essentially, given a fixed volume V, it wants us to find the height (in terms of the radius) we must have such that the surface area is minimized.

First, we know that $V = \pi r^2 h$. Solving for h gets us $h = \frac{V}{\pi r^2}$. Now, we need to get rid of V (or in other words, express it in terms of r). To do this, we need to find what r is such that the surface area is minimized.

We know that the surface area is given to be $A = 2\pi r^2 + 2\pi r h$. We can substitute $\frac{V}{\pi r^2}$ for h, in which case, we get $A = 2\pi r^2 + 2\pi r(\frac{V}{\pi r^2}) = 2\pi r^2 + \frac{2V}{r}$. We want to minimize A, in which case, we take the derivative, set it equal to 0, and we get $4\pi r - \frac{2V}{r^2}$ (remember, V is fixed, so we treat it as a constant). Setting it equal to 0, we get $4\pi r = \frac{2V}{r^2}$, so $V = 2\pi r^3$.

What we just found was what V is in terms of r when surface area is minimized. This is what we wanted, so we take this and plug it into $h = \frac{V}{\pi r^2}$. This gets us that $h = 2r$, which is the answer.

(b) We now want to find the minimum surface area in terms of V. Recall that $A = 2\pi r^2 + 2\pi r h$, in which case, we can substitute $2r$ for h (since this is what we have when A is minimized). So, we have $A = 2\pi r^2 + 2\pi r(2r) = 2\pi r^2 + 4\pi r^2 = 6\pi r^2$. We also recall that from above, we found what V is in terms of r when surface area is minimized: $V = 2\pi r^3$. Solving for r, we get that $r = (\frac{V}{2\pi})^{1/3}$. So, we substitute this into $A = 6\pi r^2$ and get $A = 6\pi(\frac{V}{2\pi})^{2/3}$, which is precisely what we wanted.